Functional lexing and parsing

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Outline of talk

- Functional vs imperative programming
- Bunch notation
- Finite state machines and regular expressions
- Context-free grammars
- Recursive descent and LL parsing
- Recursive ascent and LR parsing

Functional vs imperative programming

Functional programming:

- Based on the computation of new values by applying functions to old values
- Closer to mathematics (more conceptual)
- Typically uses recursion, lists, trees

Imperative programming:

- Based on the accretion of small changes to values
- Closer to machines (more efficient)
- Typically uses loops, arrays, frequently-modified variables

Lexing and parsing

- Major results worked out in ’60’s
- Computers were slow, memory and disk space very limited
- Dominant programming languages were low-level and imperative
- Things have changed (somewhat)
- But we still teach highly-optimized low-level algorithms!
- Now that we’re not scared of functions, recursion, lists, trees. . .

Bunches

Bunches are a variant of sets.

Intention: to simplify notation used in algorithms.

- A singleton bunch is identified with its only element.
- Bunches are "flat".

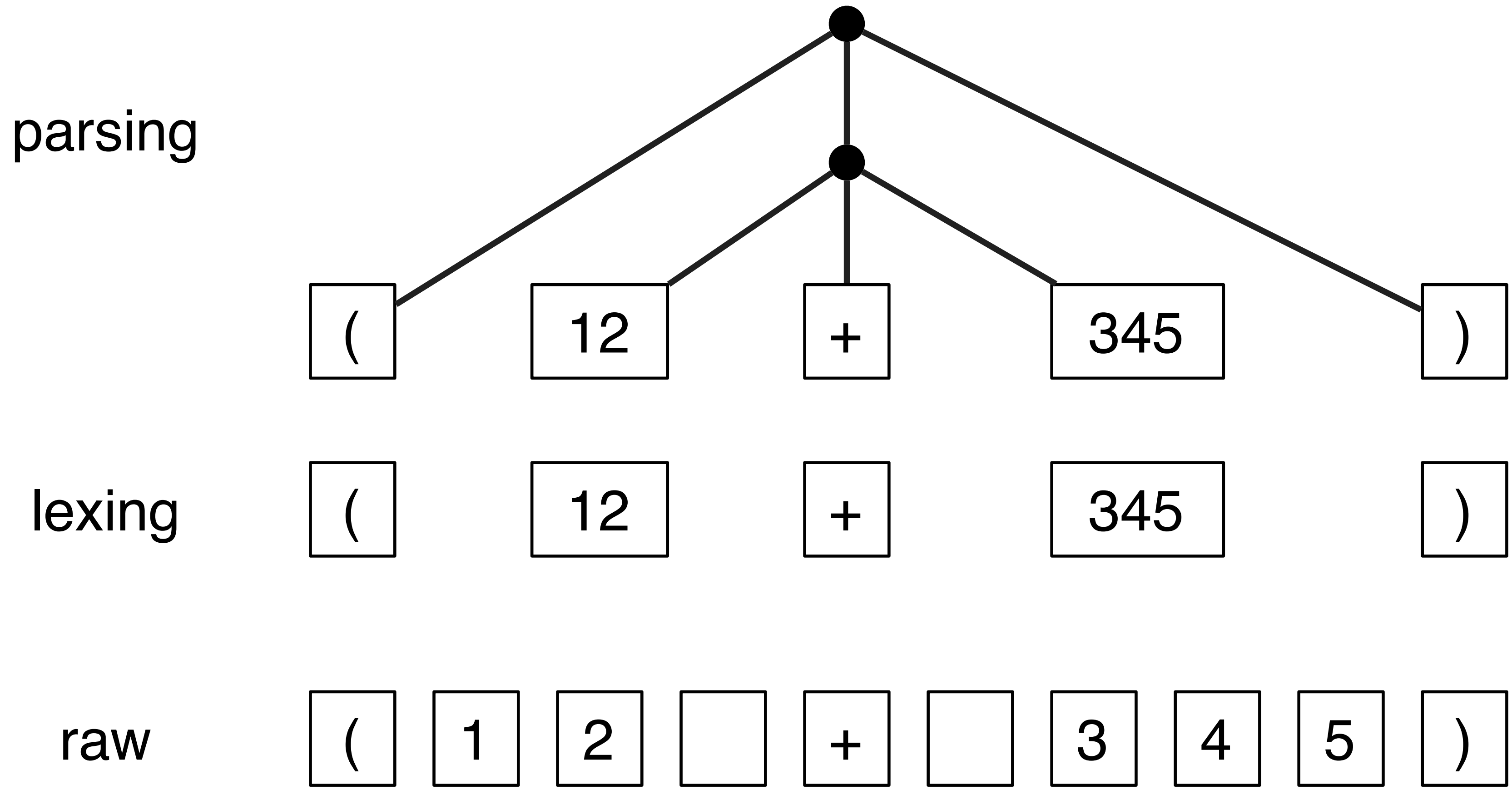
  They contain "atomic" values (not other bunches).
- Functions distribute over bunches.

  (The value of a function applied to a bunch is the bunch of the function applied to each value in the bunch.)

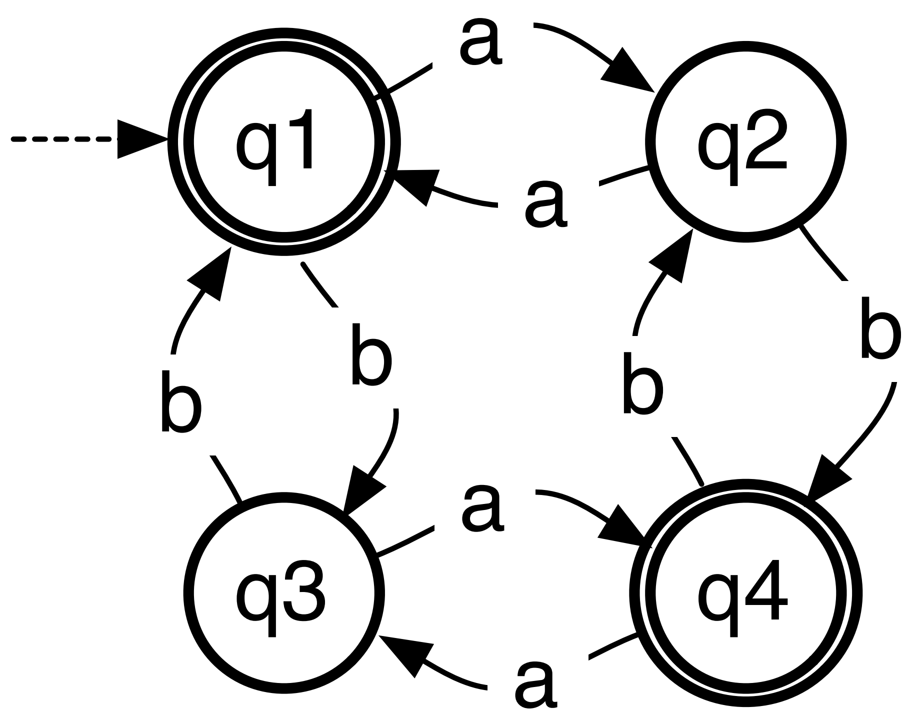
Bunch notation

- $\in$ and $\subseteq$ subsumed in $\longleftarrow$
- For union: use $\mid$ and ,
- Guards: $P \triangleright x$ means **if** P **then** x **else** ϕ.
- Implied "there exists" in guards;
  Instead of $f(x, y) = \{A(x, y) \mid \exists z\ P(x, y, z)\}$
  we write $P(x, y, z) \triangleright A(x, y)$.

Lexing and parsing



Finite state machines (FSMs)



Formally, a finite state machine M is:

- A set of states Q;
- A set of final states F;
- A start state s;
- An alphabet Σ;
- A transition function $\delta : Q \times \Sigma \rightarrow Q$.

The language L accepted by M is a subset of Σ^* (strings over the alphabet Σ).

We define a function $[q] : \Sigma^* \longrightarrow Q$ for each state q in Q.

$$[q](\sigma) = \begin{cases} q & \sigma = \epsilon \text{ (empty string)} \\ [\delta(q, \mathit{first}(\sigma))](\mathit{rest}(\sigma)) & \text{otherwise} \end{cases}$$

$$\sigma \leftarrow L \;\equiv\; [s](\sigma) \leftarrow F$$

“Interpreted”: implement $[\;]$ as a function of two arguments (q, σ)

“Compiled”: implement each $[q]$ as a separate function

```
;; "interpreted"
(define (run q sigma)
  (cond
    [(empty? q) q]
    [else (run (delta q (first sigma))
               (rest sigma))]))
```

```
// "interpreted"
q = s;
c = getchar();
while (c != EOF) {
    q = delta(q,c);
    c = getchar();
}
```

```
(define machine
  (local [(define (q1 sigma)
            (cond
              [(empty? sigma) true] ;; final state
              [else
                (case (first sigma)
                  [(a) (q2 (rest stream))]
                  [(b) (q3 (rest stream))]
                  [else false])]))
        (define (q2 sigma) . . . ) . . . ] ;; tedious repetition omitted
    q1))
```

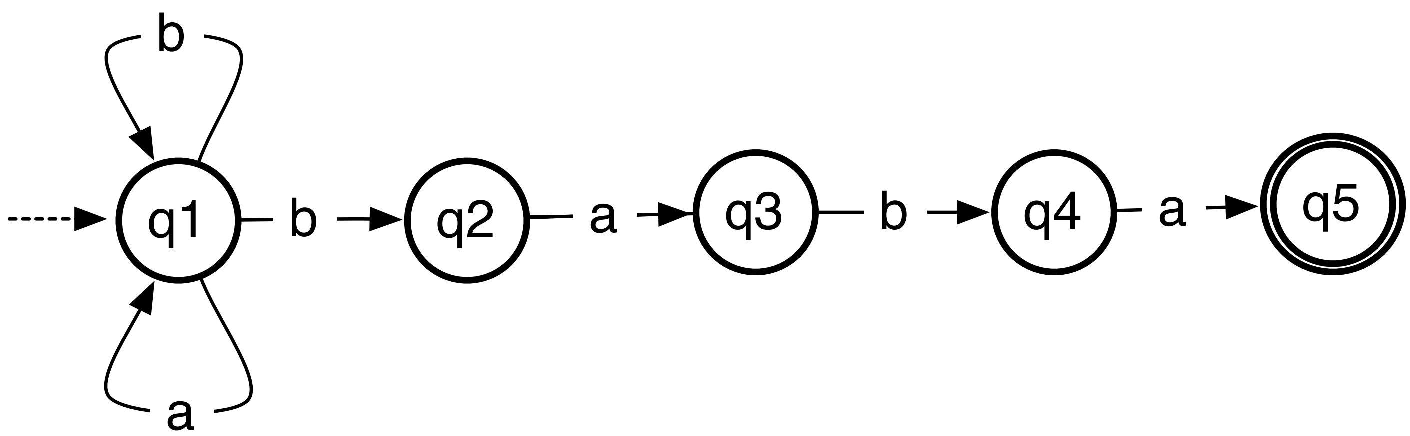
Desired syntax:

```
(define machine
  (automaton q1
    (q1 true  : (a -> q2)
                (b -> q3))
    (q2 false : (a -> q1)
                (b -> q4))
    (q3 false : (a -> q4)
                (b -> q1))
    (q4 true  : (a -> q3)
                (b -> q2))))
```

Using a macro (no omissions):

```
(define-syntax automaton
  (syntax-rules (: ->)
    [(_ init-state (state : result (symbol -> next) ...) ...)
      (local [(define (state sigma)
                  (cond
                    [(empty? sigma) result]
                    [else
                      (case (first sigma)
                        [(symbol) (next (rest sigma))] ...
                        [else false])])) ... ]
         init-state)]))
```

Nondeterministic finite state machines (NFSM)



Change: make δ a set-valued (or bunch-valued function).

Example: $\delta(q1, b) = q1, q2$.

In our definitions, we view q as a bunch (no changes needed).

We must look for a final state in the final bunch.

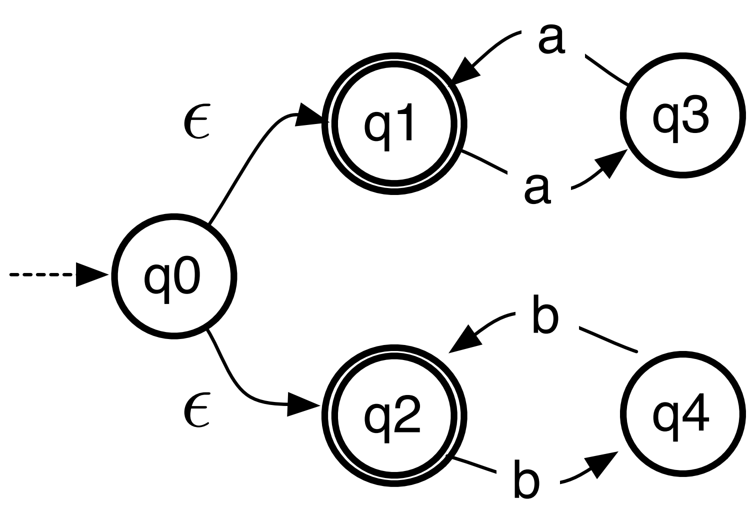
$$[q](\sigma) = \begin{cases} q & \sigma = \epsilon \\ [\delta(q, \mathit{first}(\sigma))](\mathit{rest}(\sigma)) & \text{otherwise} \end{cases}$$

$$\sigma \leftarrow L \;\equiv\; [s](\sigma) \cap F \text{ is nonempty}$$

“Interpreted”: Classical “simulation” of an NFA.

“Compiled”: the subset construction (NFA to DFA).

Adding ϵ-transitions



Change: add $eps(q) \leftarrow Q$

Example: $eps(q0) = q1, q2$.

To fix our definitions: define the $reach$ function.

$$reach(q) = q \mid eps(reach(q))$$

$$[q](\sigma) = \begin{cases} q & \sigma = \epsilon \\ [\delta(reach(q), first(\sigma))](rest(\sigma)) & \text{otherwise} \end{cases}$$

$$\sigma \leftarrow L \;\equiv\; [s](\sigma) \cap F \text{ is nonempty}$$

But how do we compute $reach(q)$?

Fixed-point computation

$reach(q)$ is a solution of $b = f(b)$ for:

$$f(b) = q \mid b \mid eps(b)$$

Here f is monotone: if $x \leftarrow y$, then $f(x) \leftarrow f(y)$.

One solution is

$$b = f(\phi) \mid f(f(\phi)) \mid f(f(f(\phi))) \ldots = \bigcup_{i=0}^{\infty} f^{(i)}(\phi)$$

This is the smallest solution, and it is a finite computation if the size of b is bounded. We say b is a **fixed point** of f.

Regular expressions (REs)

Examples: $(a+b)^*baba$, $1(0+1)^*$.

A RE R is either ϕ or ϵ or t ($t \leftarrow \Sigma$) or R_1R_2 or $R_1 + R_2$ or R_1^*.

$$
\begin{aligned}
L(R) = \;& R = \phi \triangleright \phi \mid R = \epsilon \triangleright \epsilon \mid R = t \triangleright t \\
& \mid R = R_1R_2 \triangleright L(R_1)L(R_2) \\
& \mid R = R_1 + R_2 \triangleright (L(R_1) \mid L(R_2)) \\
& \mid R = R_1^* \triangleright L(R_1)^*
\end{aligned}
$$

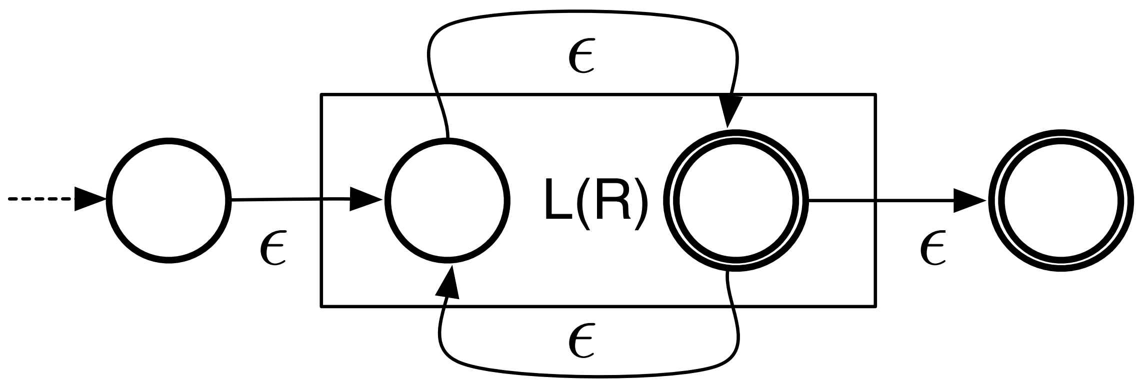
where

$$
\begin{aligned}
L_1L_2 = \;& x_1 \leftarrow L_1 \wedge x_2 \leftarrow L_2 \triangleright x_1x_2 \\
L^* = \;& x \leftarrow L \wedge y \leftarrow L^* \triangleright xy \text{ (fixed-point)}
\end{aligned}
$$

A RE has a recursive structure that is easily represented by a tree.

Various simplifications ($\epsilon R = R$, $\phi + R = R$, $\epsilon^* = \epsilon$) can be implemented with "smart constructors".

The traditional approach: convert an RE to an ϵ-NFA, then to an NFA, then to a DFA (or simulate the NFA).



Problem: adding some operators (e.g. $\neg$) becomes difficult.

A functional approach to REs

Goal: define the RE-valued $[R](\sigma)$, with specification $\gamma \leftarrow L([R](\sigma))$ if and only if $\sigma\gamma \leftarrow L(R)$.

First: define the RE-valued $nbl(R)$ (meaning "R is nullable").

$$nbl(R) \equiv \begin{cases} \epsilon & \epsilon \leftarrow L(R) \\ \phi & \text{otherwise} \end{cases}$$

$$\begin{aligned} nbl(R) = \quad & R = \phi \triangleright \phi \mid R = \epsilon \triangleright \epsilon \mid R = t \triangleright \phi \\ & \mid R = R_1 R_2 \triangleright nbl(R_1) nbl(R_2) \\ & \mid R = R_1 + R_2 \triangleright nbl(R_1) + nbl(R_2) \\ & \mid R = R_1^* \triangleright \epsilon \end{aligned}$$

Next: define $\partial_t(R)$, the "derivative with respect to t of R", with specification $t\alpha \leftarrow L(R)$ if and only if $\alpha \leftarrow L(\partial_t(R))$.

To compute $\partial_t(R)$:

$$
\begin{aligned}
\partial_t(R) = \quad & R = \phi \triangleright \phi \mid R = \epsilon \triangleright \phi \mid R = t \triangleright \epsilon \mid R = t' \triangleright \phi \\
& \mid R = R_1 R_2 \triangleright \partial_t(R_1) R_2 + nbl(R_1)\partial_t(R_2) \\
& \mid R = R_1 + R_2 \triangleright \partial_t(R_1) + \partial_t(R_2) \\
& \mid R = R_1^* \triangleright \partial_t(R_1) R_1
\end{aligned}
$$

Example: $\partial_b((a+b)^* baba) = aba + (a+b)^* baba$.

Now it is easy to define $[R](\sigma)$.

$$[R](\sigma) = \begin{cases} R & \sigma = \epsilon \\ [\partial_{\mathit{first}(\sigma)}(R)](\mathit{rest}(\sigma)) & \text{otherwise} \end{cases}$$

$$\sigma \leftarrow L(R) \;\equiv\; nbl([R](\sigma)) = \epsilon$$

“Interpreted”: structural recursion on R, tail recursion on σ.

“Compiled”: another DFA construction.

Adding new operators is much simpler.

Context-free grammars

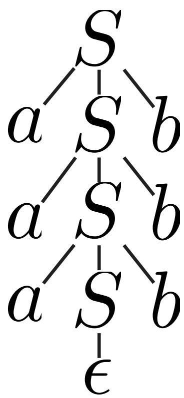
A grammar G consists of:

- A set of terminals T (here $a, b, c \ldots$);
- A set of nonterminals N (here $A, B, C \ldots$ or $\langle X \rangle$);

  (here, strings of the above are $\alpha, \beta, \ldots$)
- A set of rules R (e.g. $A \rightarrow aBa$);
- A starting nonterminal S.

Example grammar: $S \rightarrow aSb$, $S \rightarrow \epsilon$.

Rewriting: $S \rightarrow aSb \rightarrow aaSbb \rightarrow aaaSbbb \rightarrow aaabbb$.



Recognition: can a given string be produced by the grammar?

Parsing: produce the parse tree[s] for a given string.

Traditionally: a rewriting step is $\beta A\gamma \to \beta\alpha\gamma$ where $A \to \alpha$ is a rule.

$$\alpha \xrightarrow{*} \beta \;\equiv\; (\alpha = \beta) \vee (\alpha \xrightarrow{+} \beta)$$

$$\alpha \xrightarrow{+} \beta \;\equiv\; \exists\gamma(\alpha \to \gamma \wedge \gamma \xrightarrow{*} \beta)$$

$$L_G = \{\alpha \in T^* \mid S \xrightarrow{*} \alpha\}$$

Nontraditionally: define $L_G(\,\bullet\,)$ on strings from $(T|N)^*$.

$$L_G(t) = t$$

$$L_G(\epsilon) = \epsilon$$

$$L_G(\alpha\beta) = L_G(\alpha)L_G(\beta)$$

For $A \leftarrow N$, $L_G(A) = (A \to \alpha) \leftarrow R \triangleright L_G(\alpha)$

These equations in the unknowns $L_G(A)$ can be solved by a (possibly infinite) fixed-point computation, and $L_G = L_G(S)$.

Grammars and state machines

We can simulate an ϵ-NFSM using a grammar.

A state q corresponds to a nonterminal $\langle q \rangle$.

The start state s yields the rule $S \longrightarrow \langle s \rangle$.

A transition $\delta(q, c) = q'$ yields the rule $\langle q \rangle \longrightarrow c\langle q' \rangle$.

An ϵ-transition $q' \leftarrow eps(q)$ yields the rule $\langle q \rangle \longrightarrow \langle q' \rangle$.

A final state f yields the rule $\langle f \rangle \longrightarrow \epsilon$.

We will be using this idea later on.

Recognition of context-free languages

We define functions $[\gamma](\,\bullet\,)$ for $\gamma \leftarrow (T|N)^*$ with the specification
$[\gamma](\sigma) \equiv (\sigma = \sigma_1\sigma_2) \wedge \sigma_1 \leftarrow L_G(\gamma) \triangleright \sigma_2$.

$$
\begin{aligned}
[\epsilon](\sigma) &= \sigma \\
[t](\sigma) &= (\mathit{first}(\sigma) = t) \triangleright \mathit{rest}(\sigma) \\
[X\beta](\sigma) &= [\beta]([X](\sigma)) \\
[A](\sigma) &= (A \rightarrow \alpha) \leftarrow R \triangleright [\alpha](\sigma)
\end{aligned}
$$

This is a **recursive descent** parser.

$\sigma \leftarrow L \equiv \epsilon \leftarrow [S](\sigma)$

Example: $S \rightarrow aSb$, $S \rightarrow \epsilon$.

$$
\begin{aligned}
[S](aabb) &= [aSb](aabb) \mid [\epsilon](aabb) \\
&= [Sb]([a](aabb)) \mid aabb \\
&= [Sb](abb) \mid aabb \\
&= \epsilon, aabb \quad \text{because:} \\
[Sb](abb) &= [b]([S](abb)) \\
&= [b]([aSb](abb) \mid [\epsilon](abb)) \\
&= [b]([Sb](bb) \mid abb) \\
&= [b]([b]([S](bb))) \\
&= [b]([b]([aSb](bb) \mid [\epsilon](bb))) \\
&= [b]([b](bb)) = \epsilon
\end{aligned}
$$

Problem: left recursion

Example: $S \rightarrow Sa$, $S \rightarrow \epsilon$.

$$[S](\sigma) = [a]([S](\sigma)) \mid [\epsilon](\sigma)$$

Solution: Rewrite the grammar to eliminate left recursion.

Problem: it's less natural.

Problem: parse trees have the "wrong shape".

Left recursion arises naturally from left-associative operators.

Example: $a + b + c + d$ means $((a + b) + c) + d$.

We will come back to this problem.

For the time being, we avoid left recursion.

Problem: running time

Recursive descent is slow for some grammars without left recursion.

Example: $S \rightarrow aSS$, $S \rightarrow \epsilon$.

Recursive descent on a string of n a's takes exponential time.

Solution: memoization.

Create a table of previously computed function values.

There are $O(1)$ function "names" (nonterminals, suffixes of rule RHSs).

There are $O(n)$ arguments (suffixes of input).

A table entry (bunch) could be of size $O(n)$, and computing it could take $O(n^2)$ time.

Time complexity $O(n^3)$, space complexity $O(n^2)$.

Problem: still too much time/space used

Idea: use the next character in the input to eliminate unnecessary recursion (perhaps to the point of eliminating bunches).

$$[A](\sigma) = (A \to \alpha) \leftarrow R \triangleright [\alpha](\sigma)$$

If nothing in $L_G(\alpha)$ starts with $\mathit{first}(\sigma)$, don't call $[\alpha]$.

Complication: what if $\sigma \leftarrow [\alpha](\sigma)$ (i.e., $\epsilon \leftarrow L_G(\alpha)$)?

Then we must check if $\mathit{first}(\sigma)$ can follow A in some rewriting of S.

As a utility predicate, we define the Boolean-valued $nbl(\alpha) \equiv \epsilon \leftarrow L_G(\alpha)$ for α a suffix of a rule RHS.

$$
\begin{aligned}
nbl(\epsilon) &= true \\
nbl(t) &= false \\
nbl(X\beta) &= nbl(X) \wedge nbl(\beta) \\
nbl(A) &= (A \to \epsilon) \leftarrow R \triangleright true \mid (A \to \alpha) \leftarrow R \triangleright nbl(\alpha)
\end{aligned}
$$

This is a finite fixed-point computation.

For use in the "first" condition, we define $\mathit{first}(\alpha) \equiv t\beta \leftarrow L_G(\alpha) \triangleright t$ for α a suffix of a rule RHS.

$$
\begin{aligned}
\mathit{first}(\epsilon) &= \phi \\
\mathit{first}(t) &= t \\
\mathit{first}(X\beta) &= \mathit{first}(X) \mid \mathit{nbl}(X) \triangleright \mathit{first}(\beta) \\
\mathit{first}(X) &= (X \rightarrow t\alpha) \leftarrow R \triangleright t \\
&\mid (X \rightarrow Y\alpha) \leftarrow R \triangleright \mathit{first}(Y)
\end{aligned}
$$

This is a finite fixed-point computation.

For use in the "follow" condition, we define $follow(X)$ for $X \leftarrow N$.

Specification:

$follow(X) \equiv \alpha X \beta \leftarrow L_G(S) \wedge \mathit{first}(\beta) \leftarrow T \triangleright \mathit{first}(\beta)$.

$$\begin{aligned} follow(X) = {} & (A \rightarrow \alpha X \beta) \leftarrow R \wedge \mathit{first}(\beta) \leftarrow T \triangleright \mathit{first}(\beta) \\ & \mid (A \rightarrow \alpha X \beta) \leftarrow R \wedge nbl(\beta) \triangleright follow(A) \end{aligned}$$

This is a finite fixed-point computation.

We are finally ready to modify our recursive descent parser.

$$
\begin{aligned}
[\epsilon](\sigma) &= \sigma \\
[t](\sigma) &= (\mathit{first}(\sigma) = t) \triangleright \mathit{rest}(\sigma) \\
[X\beta](\sigma) &= [\beta]([X](\sigma)) \\
[A](\sigma) &= (A \to \alpha) \leftarrow R \;\wedge \\
&\qquad ((\mathit{first}(\alpha) = \mathit{first}(\sigma)) \vee (\mathit{nbl}(\alpha) \wedge \mathit{first}(\sigma) \leftarrow \mathit{follow}(A))) \\
&\qquad \triangleright [\alpha](\sigma)
\end{aligned}
$$

A grammar is **LL(1)** iff this is “deterministic” (there is at most one rule making the guard true).

For **LL(k)**, we define first_k and follow_k (k symbols of lookahead).

To obtain the conventional algorithm: make the recursion stack explicit.

```
push S
while (stack nonempty) {
    if (top is terminal t) {
        if (input symbol is t) {
            pop t, consume t
        } else {
            pop A
            push RHS of rule rewriting A
        }
}
accept iff input empty
```

Before we move towards LR parsing. . .

Some alternatives:

ANTLR and LL(*) parsing

Parsing expression grammars and packrat parsing

Parser combinators

A grammar transformation

Aim: to ensure at most two nonterminals on RHS of any rule.

Idea: create new nonterminals which are **items** of the form $\langle A \to \alpha \bullet \beta \rangle$, where $(A \to \alpha\beta) \leftarrow R$.

Given a grammar G, create E_G with the following rules:

$A \to \langle A \to \ \bullet \alpha \rangle$ for $(A \to \alpha) \leftarrow R$

$\langle A \to \alpha \bullet X\beta \rangle \to X\langle A \to \alpha X \bullet \beta \rangle$ for $(A \to \alpha X \beta) \leftarrow R$

$\langle A \to \alpha \bullet \rangle \to \epsilon$ for $(A \to \alpha) \leftarrow R$

G and E_G define the same language.

Apply recursive descent to E_G.

$$
\begin{aligned}
[t](\sigma) &= (\mathit{first}(\sigma) = t) \triangleright \mathit{rest}(\sigma) \\
[A](\sigma) &= (A \to \alpha) \leftarrow R \triangleright [A \to \ \bullet \alpha](\sigma) \\
[A \to \alpha \bullet X\beta](\sigma) &= [A \to \alpha X \bullet \beta]([X](\sigma)) \\
[A \to \alpha \bullet](\sigma) &= \sigma
\end{aligned}
$$

Inline $[t]$ and $[A]$, so all remaining functions have “item names”.

$$
\begin{aligned}
[A \to \alpha \bullet t\beta](\sigma) &= (\mathit{first}(\sigma) = t) \triangleright [A \to \alpha \bullet t\beta](\mathit{rest}(\sigma)) \\
[A \to \alpha \bullet B\beta](\sigma) &= (B \to \gamma) \leftarrow R \triangleright [A \to \alpha B \bullet \beta]([B \to \ \bullet \gamma](\sigma)) \\
[A \to \alpha \bullet](\sigma) &= \sigma
\end{aligned}
$$

If we add the rule $S' \rightarrow S$ to the grammar, then
$\sigma \leftarrow L_G(S) \equiv \epsilon \leftarrow [S' \rightarrow \bullet S](\sigma)$.

This is just a variation on recursive descent.

Memoized, it is still an $O(n^3)$ algorithm.

And it still has problems with left recursion.

A better grammar transformation can deal with left recursion.

We say A is a **left corner** of α if by rewriting the leftmost symbol repeatedly, we get from α to $A\beta$.

We'll abbreviate this as $lc(A, \alpha)$.

$$lc(A, \alpha) = (A = \mathit{first}(\alpha)) \vee ((first(\alpha) \to \gamma) \leftarrow R \wedge lc(A, \gamma))$$

This is another finite fixed-point computation.

We add nonterminals of the form $\langle X, A \to \alpha \bullet \beta \rangle$, meaning, intuitively, that we've seen α, we hope to see β, and $lc(X, \beta)$.

We use this idea to create a grammar F_G equivalent to G, with rules of the five types listed on the next slide.

Type 1: $S \rightarrow \langle S \rightarrow \ \bullet \alpha \rangle$ for $(S \rightarrow \alpha) \leftarrow R$

Type 2: $\langle X, A \rightarrow \alpha \bullet X\beta \rangle \rightarrow \langle A \rightarrow \alpha X \bullet \beta \rangle$ for $(A \rightarrow \alpha X\beta) \leftarrow R$

Type 3: $\langle A \rightarrow \alpha \bullet \beta \rangle \rightarrow t\langle t, A \rightarrow \alpha \bullet \beta \rangle$ for $(A \rightarrow \alpha\beta) \leftarrow R \wedge lc(t, \beta)$.

Type 4: $\langle X, A \rightarrow \alpha \bullet \beta \rangle \rightarrow \langle B \rightarrow X \bullet \delta \rangle \langle B, A \rightarrow \alpha \bullet \beta \rangle$ for $(B \rightarrow X\delta), (A \rightarrow \alpha\beta) \leftarrow R \wedge lc(B, \beta)$.

Type 5: $\langle A \rightarrow \alpha \bullet \rangle \rightarrow \epsilon$ for $(A \rightarrow \alpha) \leftarrow R$

Claim: this is not left-recursive if G is not cyclic (we cannot rewrite A and get A) and has no ϵ-rules (that can be fixed with a sixth type of rule).

Example: $S \rightarrow Sx$, $S \rightarrow y$.

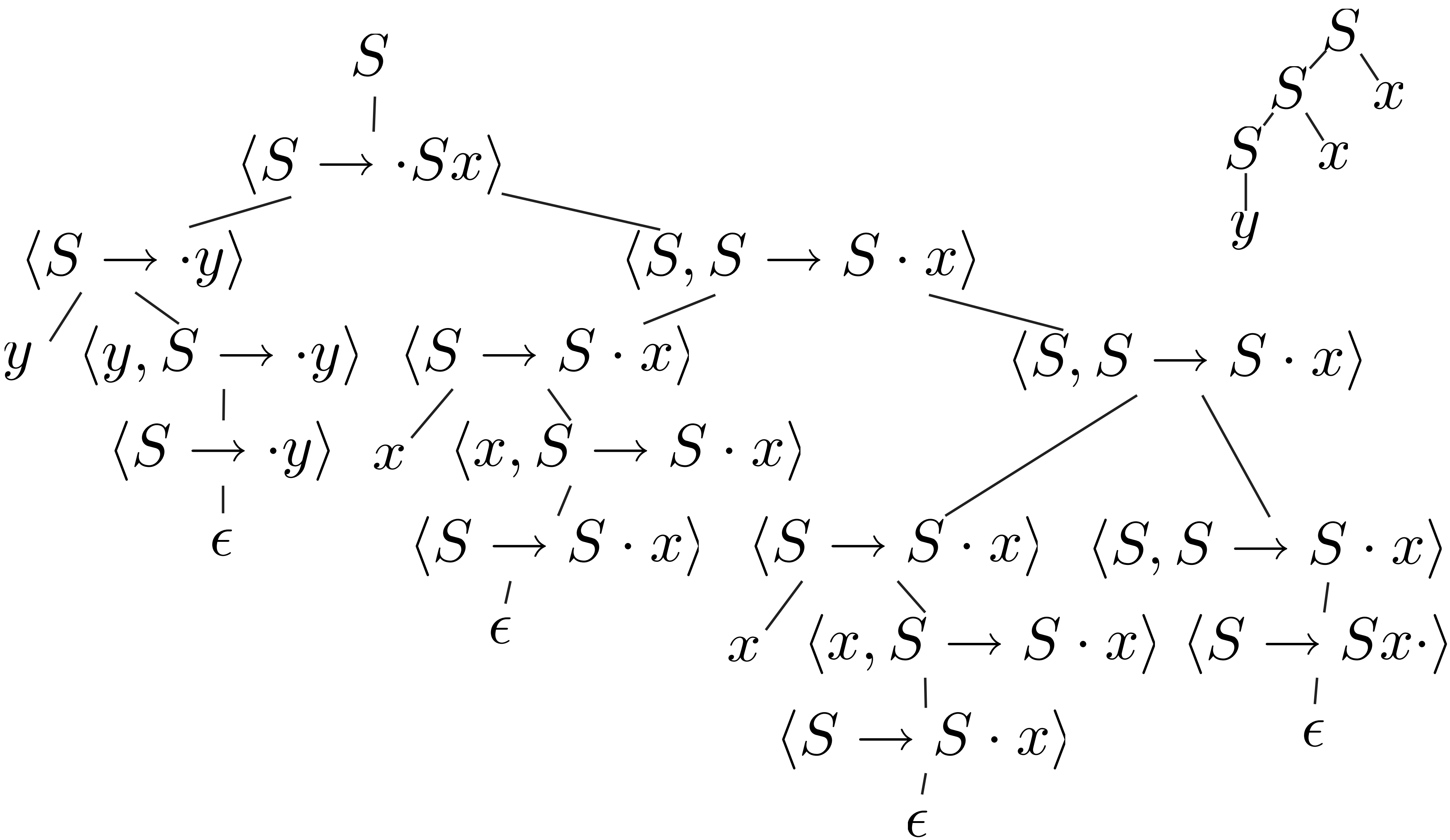
Type 1: $S \rightarrow \langle S \rightarrow \bullet Sx \rangle$, $S \rightarrow \langle S \rightarrow \bullet y \rangle$.

Type 2: $\langle S, S \rightarrow \bullet Sx \rangle \rightarrow \langle S \rightarrow S \bullet x \rangle$,
$\langle x, S \rightarrow S \bullet x \rangle \rightarrow \langle S \rightarrow Sx \bullet \rangle$, $\langle y, S \rightarrow \bullet y \rangle \rightarrow \langle S \rightarrow y \bullet \rangle$.

Type 3: $\langle S \rightarrow S \bullet x \rangle \rightarrow x \langle x, S \rightarrow S \bullet x \rangle$,
$\langle S \rightarrow \bullet y \rangle \rightarrow y \langle y, S \rightarrow y \bullet \rangle$.

Type 4: $\langle S \rightarrow \bullet Sx \rangle \rightarrow \langle S \rightarrow S \bullet x \rangle \langle S, S \rightarrow S \bullet x \rangle$,
$\langle S \rightarrow \bullet y \rangle \rightarrow \langle S \rightarrow \bullet y \rangle \langle S, S \rightarrow S \bullet x \rangle$.

Type 5: $\langle S \rightarrow Sx \bullet \rangle \rightarrow \epsilon$, $\langle S \rightarrow y \bullet \rangle \rightarrow \epsilon$.



S
$\langle S \rightarrow \cdot Sx \rangle$
$\langle S \rightarrow \cdot y \rangle$
$\langle S, S \rightarrow S \cdot x \rangle$
y
$\langle y, S \rightarrow \cdot y \rangle$
$\langle S \rightarrow S \cdot x \rangle$
$\langle S, S \rightarrow S \cdot x \rangle$
$\langle S \rightarrow \cdot y \rangle$
x
$\langle x, S \rightarrow S \cdot x \rangle$
ϵ
$\langle S \rightarrow S \cdot x \rangle$
$\langle S \rightarrow S \cdot x \rangle$
$\langle S, S \rightarrow S \cdot x \rangle$
ϵ
x
$\langle x, S \rightarrow S \cdot x \rangle$
$\langle S \rightarrow Sx \cdot \rangle$
$\langle S \rightarrow S \cdot x \rangle$
ϵ
ϵ
S
S
x
S
x
y

We apply recursive descent to F_G.

We'll have functions of the form $[A \rightarrow \alpha.\beta](\sigma)$ which, intuitively, removes from σ something obtainable by rewriting β.

We will represent the $[X, A \rightarrow \alpha \bullet \beta](\sigma)$ functions as $[\overline{A \rightarrow \alpha \bullet \beta}](X, \sigma)$.

The resulting parser is shown on the next slide.

$$[A \to \alpha \bullet \beta](\sigma) = lc(\mathit{first}(\sigma), \beta) \triangleright [\overline{A \to \alpha \bullet \beta}](\mathit{first}(\sigma)), \mathit{rest}(\sigma))$$

$$\mid lc(B, \beta) \triangleright [\overline{A \to \alpha \bullet \beta}](B, \sigma)$$

$$\mid \beta = \epsilon \triangleright \sigma$$

$$[\overline{A \to \alpha \bullet \beta}](X, \sigma) = (\beta = X\gamma) \triangleright [A \to \alpha X \bullet \gamma](\sigma)$$

$$\mid lc(B, \beta) \wedge (B \to X\delta) \leftarrow R$$

$$\triangleright [\overline{A \to \alpha \bullet \beta}](B, [B \to X \bullet \delta](\sigma))$$

This is a **recursive ascent** parser.

Memoized, the recursive ascent parser still has $O(n^3)$ time complexity and $O(n^2)$ space complexity when parsing strings of length n.

It can handle left-recursive grammars, and it can be augmented to produce a compact representation of all possible parse trees of the parsed string.

We need to add one more idea in order to design LR parsers with $O(n)$ time and space complexity (for a restricted set of grammars).

Recall our simulation of a finite-state machine by a grammar.

Let's examine some of the rules in F_G.

Type 3: $\langle A \to \alpha \bullet \beta \rangle \to t \langle t, A \to \alpha \bullet \beta \rangle$ for $(A \to \alpha\beta) \leftarrow R \wedge lc(t, \beta)$.

This looks like a simulated state transition on t.

Type 2: $\langle X, A \to \alpha \bullet X\beta \rangle \to \langle A \to \alpha X \bullet \beta \rangle$ for $(A \to \alpha X \beta) \leftarrow R$

This could be viewed as a state transition on X.

Type 4: $\langle X, A \to \alpha \bullet \beta \rangle \to \langle B \to X \bullet \delta \rangle \langle B, A \to \alpha \bullet \beta \rangle$ for $(B \to X\delta), (A \to \alpha\beta) \leftarrow R \wedge lc(B, \beta)$.

This is like an ϵ-transition from working on X to working on B.

The analogy is not perfect, but if:

- a rule is like a transition, and
- not knowing what rule to apply is like not knowing what transition to make,

then we can use a variant on our definition of the meaning of a nondeterministic finite state machine (NFSM).

We will write functions $[q]$ where q is no longer just an item, but a bunch of items.

Just as our NFSM functions could be thought of as “trying all transitions in parallel”, so our parsing functions will try all possible “transitions” defined by F_G “in parallel”.

A bunch of items is called a **state** in the classic presentation.

LR parsing

For each state q, we'll define $[q](\sigma)$ with specification

$(A \to \alpha \bullet \beta) \leftarrow q \wedge \sigma = \sigma_1\sigma_2 \wedge \sigma_1 \leftarrow L_G(\beta) \triangleright (A \to \alpha \bullet \beta, \sigma_2)$.

Here's how we recognize strings generated by our grammar:

$$\sigma \leftarrow L_G(S) \equiv (S' \to S, \epsilon) \leftarrow [S' \to \bullet S](\sigma)$$

Our "ϵ-transitions" will be:

$$\begin{aligned} eps(q) = {} & (A \to \alpha \bullet B\beta) \leftarrow q \wedge (B \to \nu) \leftarrow R \\ & \triangleright B \to \bullet \nu \end{aligned}$$

As before, $reach(q) = q \mid eps(reach(q'))$.

(This is called $predict$ in the classical presentation, and has a description in terms of left corners.)

We then get the transition function:

$$goto(q, X) = (A \to \alpha \bullet X\beta) \leftarrow reach(q) \triangleright A \to \alpha X \bullet \beta$$

This defines the **LR(0) automaton** of the grammar.

We now apply the recursive ascent idea.

We define auxiliary functions $[\overline{q}]$ with specification:

$$\begin{aligned}[\overline{q}](X, \sigma) = (A \to \alpha \bullet \beta) \leftarrow R \wedge lc(X, \beta) \wedge \\ \sigma = \sigma_1\sigma_2 \wedge \sigma_1 \leftarrow L_G(rest(\beta)) \\ \triangleright [A \to \alpha.\beta](\sigma_2)\end{aligned}$$

Working out the details, we get the **LR(0)** parser on the next slide.

$$
\begin{aligned}
[q](\sigma) = {} & [\overline{q}](\mathit{first}(\sigma), \mathit{rest}(\sigma)) \\
& \quad \mid (B \rightarrow \bullet) \leftarrow \mathit{reach}(q) \triangleright [\overline{q}](B, \sigma) \\
& \quad \mid (A \rightarrow \alpha \bullet) \leftarrow q \triangleright (A \rightarrow \alpha \bullet, \sigma) \\
[\overline{q}](X, \sigma) = {} & (A \rightarrow \alpha \bullet X\gamma) \leftarrow q \wedge \\
& \quad (A \rightarrow \alpha X \bullet \gamma, \sigma') \leftarrow [\mathit{goto}(q, X)](\sigma) \\
& \quad \triangleright (A \leftarrow \alpha X \bullet \gamma, \sigma') \\
& \quad \mid C \rightarrow \bullet X\delta \leftarrow \mathit{reach}(q) \wedge \\
& \quad (C \rightarrow X \bullet \delta, \sigma') \leftarrow [\mathit{goto}(q, X)](\sigma) \\
& \quad \triangleright [\overline{q}](C, \sigma')
\end{aligned}
$$

If $[q]$ is deterministic (single-valued) for all q, the grammar is **LR(0)**.

Possible sources of nondeterminism:

- if a state q has more than one item of the form $A \to \alpha \bullet$
  (this is a **reduce-reduce** conflict)
- if a state q has an item $A \to \alpha \bullet$ but also $goto(q, t)$ is nonempty, which will be a problem if $t = \mathit{first}(\sigma)$
  (this is a **shift-reduce** conflict)

For **LR(k)**, add lookahead k as with LL(k).

This only vaguely resembles the classical description of an LR parser.

To get the classical presentation:

- make the recursion stack explicit (the “state stack”), allowing the use of a while loop
- view the input argument as a stack (the “symbol stack”) augmented by items in the case of $[q]$ and the extra argument in the case of $[\overline{q}]$, allowing input to be read a character at a time
- implement various optimizations (e.g. items never need to be pushed onto the symbol stack)

In summary

- LR parsing is hard to understand
- It gets harder when you start from the wrong end
- There are easier lexers and parsers for learning and experiment
- A functional approach facilitates understanding
  of both lexing and parsing

References

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